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DISTORTION FUNCTIONS FOR PLANE QUASICONFORMAL MAPPINGS†

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ABSTRACT

The authors study two well-known distortion functions, $\lambda(K)$ and $\varphi_K(r)$, of the theory of plane quasiconformal mappings and obtain several new inequalities for them. The proofs make use of some properties of elliptic integrals.

1. Introduction

In the distortion theory of plane quasiconformal mappings two special functions, namely $\lambda(K)$ and $\varphi_K(r)$, play an important role. The function $\lambda(K)$, introduced by Lehto, Virtanen and Väisälä [LVV], yields a sharp upper bound for the linear dilatation of a plane quasiconformal mapping, while $\varphi_K(r)$ is the sharp upper bound in the quasiconformal version of the Schwarz lemma [LV, Theorem 3.1, p. 64]. These functions will be defined in Sections 2 and 3, respectively. We now state some of the main results of this paper.

1.1. THEOREM. *The function* $(\log \lambda(K))/(K-1)$ *is strictly decreasing from* $(1, \infty)$ *onto* (π, a), *where* $a = (4/\pi)\mathscr{K}^2(1/\sqrt{2}) = \lambda'(1) = 4.37688$ *and* $\mathscr{K}$ *is a complete elliptic integral of the first kind. In particular, for* $1 < K < \infty$, $e^{\pi(K-1)} < \lambda(K) < e^{a(K-1)}$.

1.2. THEOREM. *The function* $(\log \lambda(K))/(K - 1/K)$ *is strictly increasing from* $(1, \infty)$ *onto* (b, π), *where* $b = (2/\pi)\mathscr{K}^2(1/\sqrt{2}) = \frac{1}{2}\lambda'(1) = 2.18844$ *and* $\mathscr{K}$

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is a complete elliptic integral of the first kind. In particular, for $1 < K < \infty$, $e^{b(K-1/K)} < \lambda(K) < e^{\pi(K-1/K)}$.

1.3. THEOREM. *For* $K > 1$ *and* $0 < r < 1$,

$$\frac{2r^{1/K}}{(1+r')^{1/K}+(1-r')^{1/K}} < \varphi_K(r) < \frac{2^{2-1/K}r^{1/K}}{(1+r')^{1/K}+(1-r')^{1/K}}.$$

This last theorem improves the previously known bounds $r^{1/K} < \varphi_K(r) < 4^{1-1/K}r^{1/K}$ ([AVV2, (3.5)] and [LV, (3.6), p. 65]; cf. [AVV1, (4.11)]).

In the sequel, by $\mu(r)$ we shall mean the modulus of the Grötzsch extremal ring $B^2 \setminus [0, r]$, $0 < r < 1$, in the plane, which is given by the formula

$$\mu(r) = \frac{\pi \mathcal{K}'(r)}{2\mathcal{K}(r)}, \tag{1.4}$$

where

$$\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} dt, \qquad \mathcal{K}'(r) = \mathcal{K}(r'), \quad r' = \sqrt{1-r^2} \tag{1.5}$$

are complete elliptic integrals of the first kind ([BF, #110.06], [Bo, p. 17]), whose values are listed in standard tables (e.g. [AS], [Fr]). For later reference we also recall that the complete elliptic integrals of the second kind ([BF, #110.07], [Bo, p. 17]) are defined by

$$E(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{1/2} dt, \qquad E'(r) = E(r'), \quad r' = \sqrt{1-r^2} \tag{1.6}$$

The following useful identities are satisfied by the function μ for $0 < r < 1$ (cf. [LV, (2.7), (2.9), (2.3), pp. 60, 61]):

$$\mu(r)\mu(r') = \frac{\pi^2}{4}, \quad \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}, \quad \mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right). \tag{1.7}$$

Throughout this paper, for $t \in [0, 1]$, t' will denote $\sqrt{1-t^2}$, as in (1.5) and (1.6). When the argument of the function is clear, we shall frequently write $\mathcal{K}$, $\mathcal{K}'$ and E, E' instead of $\mathcal{K}(r)$, $\mathcal{K}'(r)$ and $E(r)$, $E'(r)$. We shall follow the relatively standard notation of [LV].

Finally, it should be pointed out that $\mu(1/\sqrt{2}) = \pi/2$ by (1.7), and hence (see (2.1) and (3.1)) the functions $\lambda(K)$ and $\varphi_K(r)$ are related by the identity

(1.8)
$$\lambda(K) = \left(\frac{\varphi_K(1/\sqrt{2})}{\varphi_{1/K}(1/\sqrt{2})}\right)^2, \qquad K > 0.$$

2. The λ-distortion function

In 1959 Lehto, Virtanen and Väisälä [LVV] (cf. [LV, pp. 80–82, 105–108]) introduced an important function

(2.1)
$$\lambda(K) = (\mu^{-1}(\pi/2K)/\mu^{-1}(\pi K/2))^2,$$

which measures the distortion of the boundary values of a K-quasiconformal self-mapping of the upper half plane preserving the point ∞. We shall use (2.1) to study some properties of μ and λ.

2.2. Proof of Theorem 1.1. First, let $r = \mu^{-1}(\pi K/2)$. Then, by [AVV1, (3.3), (3.1)], $r = (1 + \lambda(K))^{-1/2}$, $0 < r \leqq 2^{-1/2}$, or $\lambda(K) = r'^2/r^2$. Thus $(\log \lambda(K))/(K-1)$ is strictly decreasing on $(1, \infty)$ if and only if the function

$$f(r) \equiv \frac{\log(r'/r)}{\dfrac{\mathscr{K}'(r)}{\mathscr{K}(r)} - 1}$$

is strictly increasing on $(0, 1/\sqrt{2})$. Now by [AVV3, Lemma 2.7],

$$rr'^2\left(\frac{\mathscr{K}'}{\mathscr{K}} - 1\right)^2 f'(r) = -\left(\frac{\mathscr{K}'}{\mathscr{K}} - 1\right) + \frac{\pi}{2\mathscr{K}^2}\log\frac{r'}{r},$$

which is positive on $(0, 1/\sqrt{2})$ if and only if

$$g(r) \equiv \frac{2}{\pi}\,\mathscr{K}(\mathscr{K}' - \mathscr{K}) - \log(r'/r) < 0$$

on this interval. But $g(1/\sqrt{2}) = 0$, while [BF, #710.00] and Legendre's relation [BF, #110.10] give

$$\pi rr'^2 g'(r) = 4(\mathscr{K}' - \mathscr{K})(E - r'^2\mathscr{K}) > 0$$

on $(0, 1/\sqrt{2})$. Thus $g'(r) > 0$, $g(r) < 0$ for $0 < r < 1/\sqrt{2}$, so that $f'(r) > 0$ on this interval.

Finally,

$$\lim_{K\to\infty}\frac{\log\lambda(K)}{K-1} = \lim_{r\to 0}\frac{2\log(r'/r)}{2\mu(r)/\pi - 1} = \pi$$

by [AVV2, Lemma 2.6(2)], while

$$\lim_{K\to 1}\frac{\log\lambda(K)}{K-1}=\lim_{r\to 1/\sqrt{2}}\frac{2\log(r'/r)}{2\mu(r)/\pi-1}=\frac{4}{\pi}\mathscr{K}^2(1/\sqrt{2})$$

by l'Hôpital's rule and [AVV2, Lemma 2.6(2)]. The fact that $a=\lambda'(1)$ follows from l'Hôpital's rule and [AVV3, Lemma 2.7]. □

2.3. Corollary. *The function* $\log\lambda(K)$ *is concave on* $[1,\infty)$ *and satisfies the inequality* $K^{\pi\sqrt{K}}<\lambda(K)<K^{aK}$ *for* $K\in(1,\infty)$, *where a is as in Theorem* 1.1.

Proof. By elementary calculus, $(K-1)/K<\log K<(K-1)/\sqrt{K}$ for $K\in(1,\infty)$. Hence $\sqrt{K}\log K<K-1<K\log K$, and the result follows from Theorem 1.1. □

2.4. Corollary. (1) *For* $0<r<1/\sqrt{2}$,

$$\frac{\pi}{2}+\frac{\pi}{a}\log\frac{r'}{r}<\mu(r)<\frac{\pi}{2}+\log\frac{r'}{r}.$$

(2) *For* $1/\sqrt{2}<r<1$,

$$\frac{\pi^2}{4\left(\frac{\pi}{2}+\log\frac{r}{r'}\right)}<\mu(r)<\frac{\pi^2}{4\left(\frac{\pi}{2}+\frac{\pi}{a}\log\frac{r}{r'}\right)}.$$

There is equality in both (1) *and* (2) *when* $r=1/\sqrt{2}$. *Here* $a=4\kappa^2(1/\sqrt{2})$.

Proof. Statement (1) follows from Theorem 1.1 if, as usual, we set $K=2\mu(r)/\pi$, while (2) follows from (1) and (1.7). □

2.5. Proof of Theorem 1.2. As in the proof of Theorem 1.1 we set $r=\mu^{-1}(\pi K/2)=(1+\lambda(K))^{-1/2}$, $0\leqq r\leqq 2^{-1/2}$. Then the theorem is true if and only if the function

$$f(r)\equiv\frac{\log(r'/r)}{\dfrac{\mathscr{K}'(r)}{\mathscr{K}(r)}-\dfrac{\mathscr{K}(r)}{\mathscr{K}'(r)}}$$

is strictly decreasing on $(0,1/\sqrt{2})$. By [AVV3, Lemma 2.7]

$$\left(\frac{\mathscr{K}'}{\mathscr{K}}-\frac{\mathscr{K}}{\mathscr{K}'}\right)^2 rr'^2 f'(r)=-\left(\frac{\mathscr{K}'}{\mathscr{K}}-\frac{\mathscr{K}}{\mathscr{K}'}\right)+\frac{\pi}{2}\left(\frac{1}{\mathscr{K}^2}+\frac{1}{\mathscr{K}'^2}\right)\log\frac{r'}{r},$$

which is negative on $(0, 1/\sqrt{2})$ if and only if

$$g(r) \equiv \frac{2}{\pi} \mathscr{K}\mathscr{K}' \frac{\mathscr{K}'^2 - \mathscr{K}^2}{\mathscr{K}'^2 + \mathscr{K}^2} - \log \frac{r'}{r} > 0$$

there. Since $g(1/\sqrt{2}) = 0$, it is sufficient to show that $g'(r) < 0$. Next, [AVV3, Lemma 2.7] and Legendre's relation [BF] show, after some simplification, that $g'(r) < 0$ if and only if

$$\mathscr{K}'(E - r'^2\mathscr{K})(\mathscr{K}'^2 + \mathscr{K}^2) < \pi\mathscr{K}^2/2. \tag{2.6}$$

By using Legendre's relation [BF] to eliminate the factor $\pi/2$ and by performing further algebraic simplifications, we may show that (2.6) is equivalent to $h(r) < h(r')$ on $(0, 1/\sqrt{2})$, where

$$h(r) \equiv r^2\mathscr{K}'^3 \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1 - r^2\sin^2 t}}\, dt.$$

But since $r < r'$ for $r \in (0, 1/\sqrt{2})$ and since h is clearly strictly increasing, we have $h(r) < h(r')$ on $(0, 1/\sqrt{2})$, as desired. □

2.7. Corollary. *For $0 \leqq r \leqq 1/\sqrt{2}$, define the function f by $f(0) = 1$, $f(1/\sqrt{2}) = c \equiv \pi^2/(2\mathscr{K}^2(1/\sqrt{2})) = 1.43553$, and $f(r) = (\mu(r) - \mu(r'))/\log(r'/r)$ if $r \in (0, 1/\sqrt{2})$. Then f is continuous and strictly increasing. In particular, for $0 < r < 1/\sqrt{2}$,*

$$\log\left(\frac{r'}{r}\right) < \mu(r) - \mu(r') < \frac{\pi^2}{2\mathscr{K}^2(1/\sqrt{2})} \log\left(\frac{r'}{r}\right).$$

There is equality throughout when $r = 1/\sqrt{2}$.

Proof. If we set $K = 2\mu(r)/\pi$ then $1 \leqq K < \infty$, and $1/K = 2\mu(r')/\pi$ by the first identity in (1.7). Hence

$$(\log \lambda(K))/(K - 1/K) = \pi(\log(r'/r))/(\mu(r) - \mu(r')),$$

and the result follows from Theorem 1.2. □

2.8. Remark. Since $f(r) = f(r')$ for $0 \leqq r \leqq 1/\sqrt{2}$, we also obtain

$$\log\left(\frac{r}{r'}\right) < \mu(r') - \mu(r) < \frac{\pi^2}{2\mathscr{K}^2(1/\sqrt{2})} \log\left(\frac{r}{r'}\right)$$

for $1/\sqrt{2} < r < 1$.

2.9. COROLLARY. *For* $0<r<1/\sqrt{2}$,

$$\frac{1}{2}\left[\log\left(\frac{r'}{r}\right)+\sqrt{\pi^2+\left(\log\left(\frac{r'}{r}\right)\right)^2}\right]$$
$$<\mu(r)<\frac{1}{2}\left[c\log\left(\frac{r'}{r}\right)+\sqrt{\pi^2+c^2\left(\log\left(\frac{r'}{r}\right)\right)^2}\right],$$

where $c=\pi^2/(2\mathscr{K}^2(1/\sqrt{2}))$. *There is equality when* $r=1/\sqrt{2}$.

PROOF. By (1.7) we may replace $\mu(r')$ by $\pi^2/(4\mu(r))$ in 2.7, thereby obtaining

$$1<\frac{\mu^2(r)-\pi^2/4}{\mu(r)\log(r'/r)}<c=\frac{\pi^2}{2\mathscr{K}^2(1/\sqrt{2})}.$$

Hence

$$\mu^2(r)-\mu(r)\log\left(\frac{r'}{r}\right)>\frac{\pi^2}{4} \tag{2.10}$$

and

$$\mu^2(r)-c\mu(r)\log\left(\frac{r'}{r}\right)<\frac{\pi^2}{4}. \tag{2.11}$$

Solving (2.10) and (2.11) for $\mu(r)$ gives the left and right inequality, respectively. □

2.12. REMARK. For $c>0$ and $0<r<1$ define

$$G_c(r)=\frac{1}{2}\left[c\log\left(\frac{r'}{r}\right)+\sqrt{\pi^2+c^2\left(\log\left(\frac{r'}{r}\right)\right)^2}\right].$$

Then Corollary 2.9 may be written as

$$G_1(r)<\mu(r)<G_c(r)\qquad \text{for } 0<r<1\sqrt{2},$$
$$G_c(r)<\mu(r)<G_1(r)\qquad \text{for } 1/\sqrt{2}<r<1,$$

where $c=\pi^2/(2K^2(1/\sqrt{2}))$. We note that $G_c(r)G_c(r')=\pi^2/4$, so that G_c satisfies the first identity for μ in (1.7) for each $c>0$.

We next obtain a slightly improved estimate for the error term in a theorem due to Lehto, Virtanen and Väisälä [LVV, Theorem 3].

2.13. THEOREM. *For each* $K \in (1, \infty)$, $\lambda(K) = \frac{1}{16}e^{\pi K} - \frac{1}{2} + \delta(K)$, *where* $\delta(K) \in (e^{-\pi K}, 2e^{-\pi K})$.

PROOF. The upper bound was obtained in [LVV, Theorem 3].

By [LV, p. 62] we have

$$\mu(r) < \log \frac{2(1+r')}{r}.$$

Exponentiating and squaring gives

$$e^{2\mu(r)} < 4\,\frac{1+r'}{1-r'}\,;$$

so, solving for r' and using the relation $r^2 = 1 - r'^2$,

$$r < \frac{4e^{\mu(r)}}{4 + e^{2\mu(r)}}. \tag{2.14}$$

Next, for $K > 1$, if we set $r = \mu^{-1}(\pi K/2)$ then $0 < r < 1/\sqrt{2}$, and by (2.1) and (2.6) we have

$$\frac{1}{\sqrt{1+\lambda(K)}} < \frac{4e^{\pi K/2}}{4 + e^{\pi K}}.$$

Solving for $\lambda(K)$ then gives $\lambda(K) > \frac{1}{16}e^{\pi K} - \frac{1}{2} + e^{-\pi K}$. □

2.15. REMARKS. (1) One may obtain bounds for $\lambda(K)$ for $0 < K < 1$ by replacing K by $1/K$ in Theorems 1.1, 1.2 and 2.12 and appealing to the relation $\lambda(1/K) = 1/\lambda(K)$ [LV, (6.5), p. 81].

(2) The bounds in Theorem 1.1 also follow from [BAh, §4.4] since the function $P(\rho)$ studied there by Beurling and Ahlfors is the inverse function of λ (cf. [L, p. 16]).

3. The φ-distortion function

The classical Schwarz Lemma for analytic functions was generalized in 1952 by J. Hersch and A. Pfluger [HP] to the class of quasiconformal mappings of the plane unit disk. They showed that there is a strictly increasing distortion function $\varphi_K = \varphi_{K,2} : (0, 1) \to (0, 1)$ such that $|f(z)| \leqq \varphi_K(|z|)$ for each K-quasiconformal mapping of the unit disk B^2 into itself with $f(0) = 0$. This distortion function is defined by

$$\varphi_K(r) = \mu^{-1}\left(\frac{1}{K}\,\mu(r)\right). \tag{3.1}$$

For each $K \geqq 1$, the function $\varphi_K(r)$ provides a sharp upper bound for $|f(z)|$ for $|z| = r$ [LV]. For $K \in (0, 1)$ we define $\varphi_K(r)$ by (3.1), and we also define $\varphi_K(0) = 0$ and $\varphi_K(1) = 1$. Since μ is strictly decreasing on $[0, 1]$, it follows that $\varphi_K(r) \geqq r$ for $K \geqq 1$ and $\varphi_K(r) \leqq r$ for $0 < K \leqq 1$, with equality in each case if and only if $K = 1$. Clearly $\varphi_K^{-1} = \varphi_{1/K}$. In 1960 the explicit estimate

$$\varphi_K(r) < 4^{1-1/K} r^{1/K} \tag{3.2}$$

for $\varphi_K(r)$, $K \geqq 1$, $0 < r < 1$, was obtained by Wang ([W], [Hü], [LV, (3.6), p. 65]). This inequality is asymptotically sharp in the sense that $\lim_{r \to 0} r^{-1/K}\varphi_K(r) = 4^{1-1/K}$. By [AVV2, Theorem 3.4] we know that $1 < r^{-1/K}\varphi_K(r) < 4^{1-1/K}$ for each $r \in (0, 1)$, while [AVV1, Theorem 4.10] gives another pair of bounds for $\varphi_K(r)$. In this section we obtain additional inequalities and identities satisfied by this function.

3.3. THEOREM. *For $K > 0$ and $r, s \in [0, 1]$, we have*

$$\varphi_K^2(r) + \varphi_{1/K}^2(s) = 1 \Leftrightarrow r^2 + s^2 = 1, \tag{3.4}$$

$$\varphi_{1/K}(s) = (1 - \varphi_K(r))/(1 + \varphi_K(r)) \Leftrightarrow s = (1 - r)/(1 + r), \tag{3.5}$$

and

$$\varphi_K(s) = 2(\varphi_K(r))^{1/2}/(1 + \varphi_K(r)) \Leftrightarrow s = 2r^{1/2}/(1 + r). \tag{3.6}$$

PROOF. These follow from (1.7) and the fact that μ is a one-to-one function. □

3.7. THEOREM. *For $K > 0$ and $r \in [0, 1]$,*

$$\varphi_{2K}(r) = \varphi_K(2\sqrt{r}/(1 + r)), \tag{3.8}$$

$$\varphi_{2K}((1 - r)/(1 + r)) = \varphi_K(\sqrt{1 - r^2}), \tag{3.9}$$

$$\varphi_{1/K}\left(\left(\frac{1 - r}{1 + r}\right)^2\right) = \frac{1 - \varphi_{2K}(r^2)}{1 + \varphi_{2K}(r^2)}. \tag{3.10}$$

PROOF. If we divide both sides of the third identity in (1.7) by $2K$ and then apply μ^{-1} to both sides, we achieve (3.8), while (3.9) follows if we replace r by $(1 - r)/(1 + r)$ in (3.8).

Next, by (3.4), the definition of φ_K, and the third identity in (1.7) we have

$$K\mu\left(\left(\frac{1-r}{1+r}\right)^2\right)=\frac{K\pi^2}{\mu(r^2)}=\frac{\pi^2}{2\mu(\varphi_{2K}(r^2))}=\mu\left(\frac{1-\varphi_{2K}(r^2)}{1+\varphi_{2K}(r^2)}\right);$$

taking μ^{-1} of both sides then gives (3.10). □

3.11. REMARKS. (1) The relation (3.5) was proved by D. Ghisa [G].

(2) From (3.4) and the fact that $\varphi_2(r)=2\sqrt{r}/(1+r)$ [LV, p. 64] it follows easily that $\varphi_{1/2}(r)=(1-r')/(1+r')$, $0<r<1$.

(3) For n-dimensional quasiconformal mappings see [Vu]. The n-dimensional analogue of (3.4) is false; that is, for $n\geqq 3$ it is not true that $\varphi_{K,n}^2(r)+\varphi_{1/K,n}^2(r')\equiv 1$, where $\varphi_{K,n}(r)$ is the n-dimensional counterpart of $\varphi_K(r)$ (cf. [Vu, 5.61, (7.44)]. Otherwise, for $K>1$ by [AVV2, (3.6)] we would have $r'^{-k}\varphi_{1/K,n}(r')>\lambda_n^{1-1/\alpha}r'^{-K+1/\alpha}\to\infty$ as $r\to 1$, contradicting [AVV2, (3.14)]. By symmetry, this argument also applies for $0<K<1$. Alternatively (cf. [Vu, 7.58]), it may be shown that this (false) identity is equivalent to $M_n(r)M_n(r')=$ constant, which contradicts [AVV2, Lemma 2.6(2)]. Here $M_n(r)$ is the n-dimensional analogue of $\mu(r)$ (cf. [Vu, 7.58] and [AVV2]). In particular, we see by (3.4) that for $n\geqq 3$ and $K>1$ there is no $T>1$ such that $\varphi_{K,n}(r)=\varphi_{T,2}(r)$ and $\varphi_{1/K,n}(r')=\varphi_{1/T,2}(r')$ for all $r\in(0,1)$.

3.12. LEMMA. *For $K>1$, let $s=\varphi_K(r)$. Then $g(r)\equiv s'\mathscr{K}(s)/(r'\mathscr{K}(r))$ is a strictly decreasing function from* (0, 1) *onto* (0, 1).

PROOF. Since $\mu(s)=\mu(r)/K$, by differentiation it follows from [AVV3, Lemma 2.7] and Legendre's relation [BF, #110.10] that

$$(r'\mathscr{K}(r))^2g'(r)=\frac{s'\mathscr{K}(s)}{rr'\mathscr{K}'(r)}\,[\mathscr{K}(r)E'(r)-\mathscr{K}(s)E'(s)].$$

But this expression is negative since $s>r$ and $\mathscr{K}(r)E'(r)$ is strictly increasing on (0, 1). Finally, the limits $g(0+)=1$ and $g(1-)=0$ follow from l'Hôpital's rule. □

3.13. THEOREM. *For $K>1$ and $a,b\in(0,1)$*

$$\varphi_K(ab)<\varphi_K(a)\varphi_K(b).$$

PROOF. For convenience we let φ_K be denoted by φ. Fix $a\in(0,1)$ and let $f(r)=\varphi(ar)/\varphi(r)$ for $0<r\leqq 1$. Let $ar=u$, $\varphi(u)=t$, and $\varphi(r)=s$. Then by [AVV3, Lemma 2.7] and Lemma 3.12 we have

$$\frac{Kf'(r)sr}{t} = \left(\frac{t'\mathscr{K}(t)}{u'\mathscr{K}(u)}\right)^2 - \left(\frac{s'\mathscr{K}(s)}{r'\mathscr{K}(r)}\right)^2 > 0.$$

Hence f is strictly increasing, and $f(r) < f(1) = \varphi(a)$. □

In the proof of our next functional inequality for φ_K we shall need the following two lemmas.

3.14. Lemma. *Let* $f(r) = (\log \varphi_K(r))/(\log r)$. *Then for* $K > 1$, f *is a strictly decreasing function from* $(0, 1)$ *onto* $(0, 1/K)$. *For* $0 < K < 1$, f *is strictly increasing from* $(0, 1)$ *onto* $(1/K, \infty)$.

Proof. First let $K > 1$. If we set $s = \varphi_K(r)$ then $0 < r < s < 1$ and $K = \mu(r)/\mu(s)$. By [AVV3, Lemma 2.7] and [AVV3, Lemma 2.8(6)] we have

$$r(\log r)^2 f'(r) = s'^2\mathscr{K}(s)\mathscr{K}'(s)\left[\frac{\log r}{r'^2\mathscr{K}(r)\mathscr{K}'(r)} - \frac{\log s}{s'^2\mathscr{K}(s)\mathscr{K}'(s)}\right] < 0$$

for $0 < r < 1$. We achieve the transition to $0 < K < 1$ by replacing K by $1/K$. The limit as r tends to 0 follows from l'Hôpital's rule, [AVV2, (3.7)], and the asymptotic formula in [AVV2, Corollary 3.8]. As r tends to 1, the limit follows by l'Hôpital's rule and the values of $\varphi_K'(1)$ found in [AVV2, Corollary 3.8]. □

3.15. Lemma. *Let* $f(r) = (\operatorname{artanh} \varphi_K(r))/(\operatorname{artanh} r)$. *For* $K > 1$, f *is a strictly decreasing function from* $(0, 1)$ *onto* (K, ∞).

Proof. By (3.5), $f(r) = (\log \varphi_{1/K}(s))/(\log s)$, where $s = (1 - r)/(1 + r)$. Then the result follows from Lemma 3.14. □

3.16. Corollary. *For* $K > 1$, *the function* $(\operatorname{artanh} \varphi_K(\tanh x))/x$ *is strictly decreasing from* $(0, \infty)$ *onto* (K, ∞).

Proof. Put $r = \tanh x$ in Lemma 3.15. □

A function-theoretic application of 3.15 will be given below in Remark 3.30.

3.17. Theorem. *For* $K > 1$ *and* $r, s \in (0, 1)$,

$$\varphi_K\left(\frac{r + s}{1 + rs}\right) < \frac{\varphi_K(r) + \varphi_K(s)}{1 + \varphi_K(r)\varphi_K(s)}.$$

The inequality is reversed if $0 < K < 1$.

Proof. Let $K > 1$. By Corollary 3.16 and [AVV2, Lemma 2.12] we get

$$\operatorname{artanh} \varphi_K(\tanh(x+y)) < \operatorname{artanh} \varphi_K(\tanh x) + \operatorname{artanh} \varphi_K(\tanh y).$$

Now take tanh of both sides, use the addition formula for tanh, and set $\tanh x = r$, $\tanh y = s$.

Finally, the result follows for $0 < K < 1$ when we replace K by $1/K$. □

In 1984 He proved the following result [He, Lemma 1] for $K \geqq 1$. It had been mentioned by Hübner [Hü] in 1970 without a proof. The extension to $0 < K < 1$ is a simple consequence of the fact that $\varphi_{1/K} = \varphi_K^{-1}$.

3.18. THEOREM. *For $K > 1$ the function $r^{-1/K}\varphi_K(r)$ is a strictly decreasing function of $(0, 1)$ onto $(1, 4^{1-1/K})$. For $0 < K < 1$, $r^{-1/K}\varphi_K(r)$ is a strictly increasing function of $(0, 1)$ onto $(4^{1-1/K}, 1)$.*

We now prove some consequences of He's theorem.

3.19. LEMMA. *For $K \geqq 1$ and each nonnegative integer p,*

$$\varphi_K(r)^{p+1} \leqq 4^{p(1-1/K)}\varphi_K(r^{p+1}).$$

PROOF. The proof is by induction. For $p = 0$ this is trivial. For $p = 1$, He's theorem and (3.2) imply that

$$\varphi_K(r^2) \geqq r^{1/K}\varphi_K(r) \geqq 4^{-1+1/K}\varphi_K(r)^2.$$

Now assume that the lemma is true for integers $\leqq p - 1$. Then

$$\varphi_K(r^{p+1}) \geqq r^{1/K}\varphi_K(r^p) \geqq 4^{-1+1/K}\varphi_K(r)\varphi_K(r^p) \geqq 4^{(-1+1/K)p}\varphi_K(r)^{p+1}$$

by the induction hypothesis. □

3.20. THEOREM. *For $a, b \in [0, 1]$ and $K \geqq 1$,*

$$|\varphi_K(a) - \varphi_K(b)| \leqq \varphi_K(|a-b|) \leqq 4^{1-1/K}|a-b|^{1/K}, \tag{3.21}$$

and for $0 < K \leqq 1$,

$$|\varphi_K(a) - \varphi_K(b)| \geqq \varphi_K(|a-b|) \geqq 4^{1-1/K}|a-b|^{1/K}. \tag{3.22}$$

PROOF. He's theorem implies that $r^{-1}\varphi_K(r)$ is decreasing on $(0, 1)$. Hence (3.21) follows from [AVV2, Lemma 2.12] and (3.2). Then (3.22) follows from (3.21) when we replace K by $1/K$. □

3.23. COROLLARY. *For $n = 2$, $K \geqq 1$, $a, b, x, y \in [0, 1]$, $a + b = 1$, we have*

$$\varphi_K(ax+by) \leqq \tfrac{1}{2}(ax+by)^{1/K}[(ax)^{-1/K}\varphi_K(ax) + (by)^{-1/K}\varphi_K(by)]$$

and

$$\varphi_K(ax+by) \leqq \varphi_K(ax)+\varphi_K(by) \leqq \varphi_K(a)\varphi_K(x)+\varphi_K(b)\varphi_K(y).$$

These inequalities are sharp for $K=1$.

PROOF. These follow from He's theorem and Theorem 3.13. □

3.24. LEMMA. *For* $K \geqq 1$ *and* $a, b \in (0, 1)$,

$$\varphi_K(ab) \geqq \max\{b^{1/K}\varphi_K(a), a^{1/K}\varphi_K(b)\}.$$

PROOF. By He's theorem,

$$(ab)^{-1/K}\varphi_K(ab) \geqq \max\{a^{-1/K}\varphi_K(a), b^{-1/K}\varphi_K(b)\},$$

and the inequality follows. □

3.25. COROLLARY. *For* $K \geqq 1$ *and* $a, b \in (0, 1)$,

$$\varphi_K(a)\varphi_K(b) \leqq 4^{1-1/K}\varphi_K(ab).$$

PROOF. By Lemma 3.24 we have

$$(ab)^{1/K}\varphi_K(a)\varphi_K(b) \leqq (\varphi_K(ab))^2$$

and

$$(ab)^{1/K} \geqq 4^{-1+1/K}\varphi_K(ab).$$ □

3.26. THEOREM. *If* $K \geqq 1$, $p \geqq 1$, *and* $x > 0$, *then*

$$\varphi_K(\tanh px) \leqq p^{1/K}\varphi_K(\tanh x).$$

PROOF. By He's theorem,

$$(\tanh px)^{-1/K}\varphi_K(\tanh px) \leqq (\tanh x)^{-1/K}\varphi_K(\tanh x).$$

But it is easy to show by differentiation that $\tanh px \leqq p \tanh x$, and the theorem follows. □

3.27. THEOREM. *If* $K > 1$, *then* $\varphi_K(r)$ *is strictly concave on* $[0, 1]$ *and* $\varphi_K'(0) = \infty$, $\varphi_K'(1) = 0$. *If* $0 < K < 1$, *then* $\varphi_K(r)$ *is strictly convex on* $[0, 1]$ *and* $\varphi_K'(0) = 0$, $\varphi_K'(1) = \infty$.

PROOF. Let $K > 1$, $0 < r < 1$, and $s = \varphi_K(r)$. Then $\mu(s) = \mu(r)/K$ and, by [AVV3, Lemma 2.7],

$$\frac{ds}{dr} = \frac{1}{K}\frac{s}{r}\left(\frac{s'\mathscr{K}(s)}{r'\mathscr{K}(r)}\right)^2.$$

He's theorem implies that s/r is strictly decreasing as a function of r, while $g(r) \equiv s'\mathscr{K}(s)/(r',\mathscr{K}(r))$ is strictly decreasing by Lemma 3.12. Hence the derivative of $\varphi_K(r)$ is a strictly decreasing function of r if $K > 1$. The strict convexity of φ_K when $0 < K < 1$ now follows from the fact that $\varphi_{1/K}(r) = \varphi_K^{-1}(r)$.

The derivatives $\varphi_K'(0)$ and $\varphi_K'(1)$ for each $K > 0$ were obtained in [AVV2, Corollary 3.8]. □

3.28. THEOREM. *For* $0 \leqq r \leqq 1$,

$$\varphi_K(r) \leqq \begin{cases} 4^{1/K}(1+r)^{-2/K}r^{1/K}, & 1 \leqq K \leqq 2, \\ 4^{1-1/K}(1+r)^{-2/K}r^{1/K}, & K \geqq 2. \end{cases}$$

PROOF. If $K \in [1, 2]$, then by (3.11)(2) and [AVV1, (4.12)],

$$\varphi_K(r) = \varphi_{K/2}(\varphi_2(r)) \leqq \varphi_2(r)^{2/K} = 4^{1/K}(1+r)^{-2/K}r^{1/K}.$$

The proof for $K \geqq 2$ is similar, except that we use [AVV1, (4.11)]. □

3.29. PROOF OF THEOREM 1.3. The first inequality follows from [AVV2, Theorem 3.4]. For the second inequality we assume first that $s \in (0, 1)$ is so small that $4^{1-1/K}s^{1/K}$ is less than 1. Then by (3.2) and (3.1) we have

$$\frac{1}{K}\mu(s) \geqq \mu(4^{1-1/K}s^{1/K}),$$

and it follows from the third identity in (1.7) that

$$\mu\left(\frac{2\sqrt{s}}{1+s}\right) \geqq K\mu\left(\frac{2^{2-1/K}s^{1/2K}}{1+4^{1-1/K}s^{1/K}}\right).$$

If we set $r = 2\sqrt{s}/(1+s)$ and use (3.1), then since $s = (1-r')/(1+r')$ the latter inequality reduces to

$$\varphi_K(r) \leqq \frac{2^{2-1/K}r^{1/K}}{(1+r')^{1/K} + 4^{1-1/K}(1-r')^{1/K}},$$

which implies our assertion.

Next, suppose $4^{1-1/K}s^{1/K} \geqq 1$, and set $r = 2\sqrt{s}/(1+s)$. Then $s = (1-r')/(1+r')$, whence

$$4^{1-1/K}\left(\frac{1-r'}{1+r'}\right)^{1/K} \geqq 1.$$

Taking square roots and multiplying both sides by $(1+r')^{1/K}$ we obtain $2^{1-1/K}r^{1/K} \geqq (1+r')^{1/K}$, from which it follows that

$$\frac{2^{2-1/K}r^{1/K}}{(1+r')^{1/K}+(1-r')^{1/K}} \geqq \frac{2(1+r')^{1/K}}{(1+r')^{1/K}+(1-r')^{1/K}} \geqq 1 > \varphi_K(r). \qquad \square$$

3.30. Remark. Lemma 3.15 has the following application to the quasiconformal Schwarz lemma (cf. [LV, Theorem 3.1, p. 64]). Let ρ be the hyperbolic metric of the unit disk B^2, and let $f: B^2 \to B^2 \subset B^2$ (cf. [Vu, (2.18)])be K-quasiconformal. Then

$$\tanh\frac{\rho(f(x),f(y))}{2} \leqq \varphi_K\left(\tanh\frac{\rho(x,y)}{2}\right)$$

for all $x, y \in B^2$. Since ρ is a metric it follows, for instance, from [AVV2, 4.7(1)] that $\tanh \rho(x,y)/2$ is also a metric. Next, solving for $\rho(f(x), f(y))$ we obtain

$$\rho(f(x),f(y)) \leqq 2 \operatorname{artanh} \varphi_K\left(\tanh\frac{\rho(x,y)}{2}\right). \tag{3.31}$$

Now, by the monotone property 3.15 and [AVV2, 4.7(1)] it follows that the right side of (3.31) is a metric.

3.32. Remark. In 1970 O. Hübner [Hü] showed that

$$\varphi_K(r) \leqq r \exp\left[\frac{2}{\pi}\left(1-\frac{1}{K}\right) r'^2 \mathscr{K}(r)\mathscr{K}'(r)\right] \leqq 4^{1-1/K}r^{1/K}$$

when $K \geqq 1$, $0 \leqq r \leqq 1$, improving (3.2). We now show that Hübner's bound is best possible in a certain sense. Suppose

$$\varphi_K(r) \leqq r \exp\left[\frac{2}{\pi}\left(1-\frac{1}{K}\right) r'^{2c} \mathscr{K}(r)\mathscr{K}'(r)\right]$$

for some $c > 1$. Then letting K tend to ∞ and taking logarithms, we get

$$\frac{\log \frac{1}{r}}{r'^{c+1}} \leqq \frac{2}{\pi} r'^{c-1} \mathscr{K}(r)\mathscr{K}'(r).$$

But [AVV2, Lemma 2.6(2)] implies that the left side tends to ∞ as r tends to 1, while the right side tends to 0 by [BF, #112.01, 111.02], and we arrive at a contradiction.

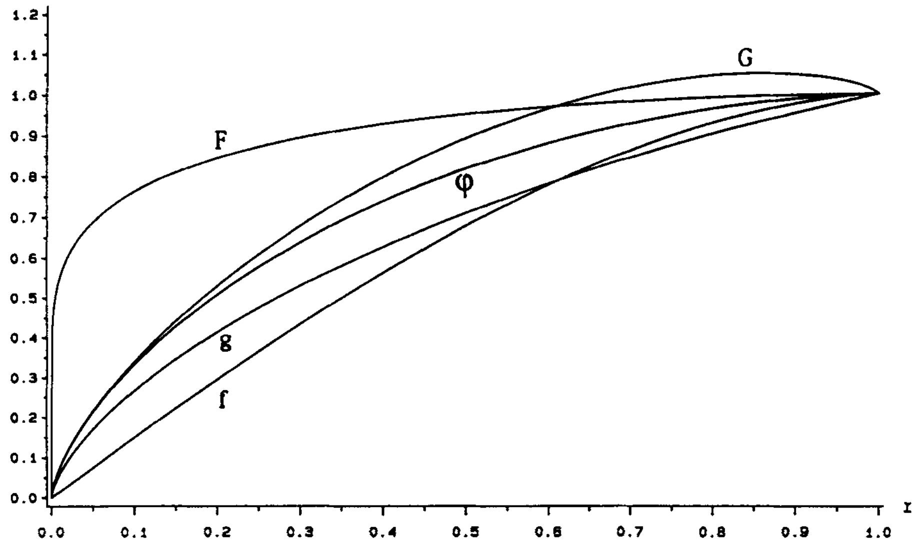

Fig. 1. $\varphi(r)=\varphi_{3/2}(r)$, $0<r<1$, and bounds thereof.

Lower: $f(r)=\tanh(\tfrac{2}{3}\operatorname{artanh} r)$, $g(r)=\dfrac{2r^{2/3}}{(1+r')^{2/3}+(1-r')^{2/3}}$.

Upper: $F(r)=\tanh\left(\dfrac{3\pi^2}{8\mu(r)}\right)$, $G(r)=r\exp\left[\dfrac{2r'^2}{3\pi}\mathscr{K}(r)\mathscr{K}'(r)\right]$.

In conclusion we display in Fig. 1 the graph of the distortion function $\varphi_{3/2}(r)$, obtained by linear interpolation from a table of values of μ that we had constructed by the arithmetic-geometric mean iteration of Gauss [BB] (cf. [AS]). Then graphs of upper and lower bounds for $\varphi_{3/2}(r)$ are shown for comparison. In particular, the functions f, F are the bounds in [AVV1, (4.5)], the function g is the lower bound in Theorem 1.3, and G is Hübner's upper bound in 3.30.

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